ASCHAM SCHOOL

YEAR 12
MATHEMATICS EXTENSION 1
TRIAL EXAMINATION 2012

GENERAL INSTRUCTIONS

5 minutes reading time
working time 2 hours
use black or blue pen
a table of standard integral is provided
approved calculators and templates may be used.

Total Marks - 70

Section 1 – MULTIPLE CHOICE (1 mark each)

- Attempt Questions 1-10
- Allow 15 minutes
- Answers on the sheet provided at the back of exam. Write your name/NUMBER, teacher's name

Section 2– Question 11 – 14 (15 marks each)

- Allow 1hour 45 minutes
- Start each question in a new booklet.
- If you use a second booklet for a question, place it inside the first.
- Write your name/NUMBER, teacher's name and question number on each booklet.

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Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section 1 **Multiple choice** **(1 mark each)**

(Mark the correct answer on the sheet provided)

1. A particle moves in a straight line and its position at any time t is given by $x = 3\cos 2t$. The motion is simple harmonic. What is the greatest speed?

(A) 3

(B) –6

(C) 2

(D) 6

2. The number N of animals in a population is increasing. At time t years the population is given by $N = 100 + Ae^{kt}$ for constants $A > 0$ and $k > 0$. Which of the following is the correct differential equation?

(A) $\dfrac{dN}{dt} = k(N - 100)$

(B) $\dfrac{dN}{dt} = -k(N + 100)$

(C) $\dfrac{dN}{dt} = -k(N - 100)$

(D) $\dfrac{dN}{dt} = k(N + 100)$

3. $\dfrac{d}{dx}(\sec\dfrac{x}{2})$ is

(A) $\sec\dfrac{x}{2}\tan\dfrac{x}{2}$

(B) $2\sec\dfrac{x}{2}\tan\dfrac{x}{2}$

(C) $\dfrac{1}{2}\sec\dfrac{x}{2}\tan\dfrac{x}{2}$

(D) $\dfrac{1}{2}\tan^2\dfrac{x}{2}$

4. The speed v (m/s) of a particle moving in a straight line is given by $v^2 = 6 + 4x - 2x^2$, where its displacement from a fixed point O is x m. The motion is simple harmonic. What is the centre of the motion?

(A) $x = -2$

(B) $x = -1$

(C) $x = 1$

(D) $x = 2$

5. What is the domain and range of $y = \cos^{-1}(\frac{3x}{2})$?

(A) Domain: $-\frac{2}{3} \le x \le \frac{2}{3}$. Range: $0 \le y \le \pi$

(B) Domain: $-\frac{3}{2} \le x \le \frac{3}{2}$. Range: $0 \le y \le \pi$

(C) Domain: $-\frac{2}{3} \le x \le \frac{2}{3}$. Range: $-\pi \le y \le \pi$

(D) Domain: $-\frac{3}{2} \le x \le \frac{3}{2}$. Range: $-\pi \le y \le \pi$

6. An object is projected with a velocity of 30 ms^{-1} at an angle of $\tan^{-1}\frac{3}{4}$ to the horizontal. What is the initial vertical component of its velocity?

(A) 18 ms^{-1}

(B) 50 ms^{-1}

(C) $30\tan\frac{3}{4}$ ms^{-1}

(D) $30\sin\frac{3}{4}$ ms^{-1}

7. If $f(x) = e^{x+2}$ what is the inverse function $f^{-1}(x)$?

(A) $f^{-1}(x) = e^{y-2}$

(B) $f^{-1}(x) = e^{y+2}$

(C) $f^{-1}(x) = \log_e x - 2$

(D) $f^{-1}(x) = \log_e x + 2$

8. Which of the following is the correct expression for $\int \frac{dx}{\sqrt{49-x^2}}$?

(A) $-\cos^{-1}\frac{x}{7}+c$

(B) $\cos^{-1}7x+c$

(C) $-\sin^{-1}\frac{x}{7}+c$

(D) $\sin^{-1}7x+c$

9. Given that $\frac{d}{dx}\left(\sin^{-1}x+\cos^{-1}x\right)=0$, the unique value of $\sin^{-1}x+\cos^{-1}x$ is ?

(A) C , where C is a constant

(B) 0

(C) $\frac{\pi}{4}$

(D) $\frac{\pi}{2}$

10. The integral of $\cos^2 2x$ is ?

(A) $\frac{\cos^3 2x}{3}+C$

(B) $\left(\frac{\cos 2x}{6}\right)^3+C$

(C) $\frac{x}{2}+\frac{\sin 4x}{8}+C$

(D) $\frac{x}{2}+\frac{\sin 2x}{4}+C$

Section 2: Answer in booklets provided

Question 11 Begin a new booklet

a) Find the product of the roots of the polynomial $P(x)=2x^4-x^3+2x-3$ **(1)**

b) Find $\frac{d}{dx}(\tan^{-1}2x^2)$. **(2)**

c) If *P* is the point (–2, 3) and *K* is the point (10, 1) find the coordinates of the point *J* which divides the interval *PK* externally in the ratio 4:3 . **(2)**

d) Find the acute angle between the lines $y=x$ and $\sqrt{3}y=-x$. **(2)**

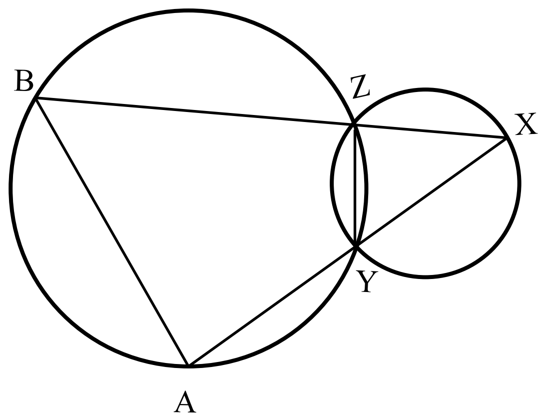
e) Solve $\frac{5}{x+2}\leq 1$. **(3)**

f) Find an expression for $\sin(\tan^{-1}x)$. **(2)**

g) Solve the equation $\sin 2x=2\sin^2 x$ for $0\leq x\leq 2\pi$. **(3)**

Question 12 **Begin a new booklet**

a)



Two circles intersect at two points Z and Y as shown in the diagram above. The diameter XY of the smaller circle produced intersects the larger circle at Y and A. The line XZ intersects the larger circle at Z and B. Prove that $\angle XAB$ is 90^0. **(2)**

b) Use the substitution $x = \frac{1}{4}(u-1)$ to evaluate **(3)**

$$\int_0^2 \frac{4x}{\sqrt{4x+1}}\,dx \ .$$

c) i) Show that $f(x) = e^x - 3x$ has a root between $x = 1$ and $x = 1.7$. **(1)**

ii) Starting with $x = 1.7$ use one application of Newton's method to find another approximation of this root correct to three significant figures. **(2)**

d) The polynomial $P(x) = x^3 + ax^2 + bx + c$ has roots 0, 3, and -3.

i) Find a, b and c . **(2)**

ii) Without using calculus, sketch the graph of $y = P(x)$. **(2)**

e) Use mathematical induction to prove that $5^n - 1$ is divisible by 4 for all positive integers, $n \geq 1$. **(3)**

Question 13 **Begin a new booklet**

a) Express $\cos x - \sin x$ in the form $A\cos(x+\alpha)$, with $A > 0$ and $0 < \alpha < \frac{\pi}{2}$. **(2)**

b) Hence find the general solution to $\cos x - \sin x = \frac{1}{\sqrt{2}}$. **(2)**

c) The metal surface of a heater is cooling after it has been switched off. The room has a constant temperature of 20° C. At a time t minutes, its temperature M decreases according to the equation $\frac{dM}{dt} = -k(M-20)$ where k is a positive constant. The initial temperature of the metal was 160° C and it cools to 100^{0} C after 5 minutes.

i) Show that $M = 20 + Ae^{-kt}$ is a solution to the differential equation, where A is a constant. **(1)**

ii) Find the values of A and k. **(2)**

iii) How long will it take for the metal surface to cool to 40° C to the nearest minute. **(2)**

d) A vitamin tablet which dissolves in water is in the shape of a cylinder. The radius of its cross section is initially 0.5 cm and the height of the tablet is always twice the radius when it dissolves. The tablet dissolves completely after 5 minutes. The rate at which the tablet dissolves is proportional to its surface area A such that $\frac{dV}{dt} = pA$ where p is a constant.

i) Show that $\frac{dr}{dt} = p$ where r is the radius of the tablet in cm at time t minutes. **(3)**

ii) Find r as a function of t. **(2)**

iii) Find the value of p. **(1)**

Question 14 Begin a new booklet

a) The acceleration of a particle *P* is given by $\ddot{x} = 8x(x^2 + 4)$ where x is the displacement of *P* from a fixed point *O* after *t* seconds. Initially the particle is at *O* and has a velocity of 8 m/s in the positive direction.

i) Show that the speed of the particle is given by $2(x^2 + 4)$ m/s . **(3)**

ii) Explain why the velocity of the particle is always positive. **(1)**

iii) Hence find the time taken for the particle to travel 2 metres from O. **(3)**

b) The two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are on the parabola $x^2 = 4ay$. The equation of the tangent at P is given by $y = px - ap^2$ (do not prove this).

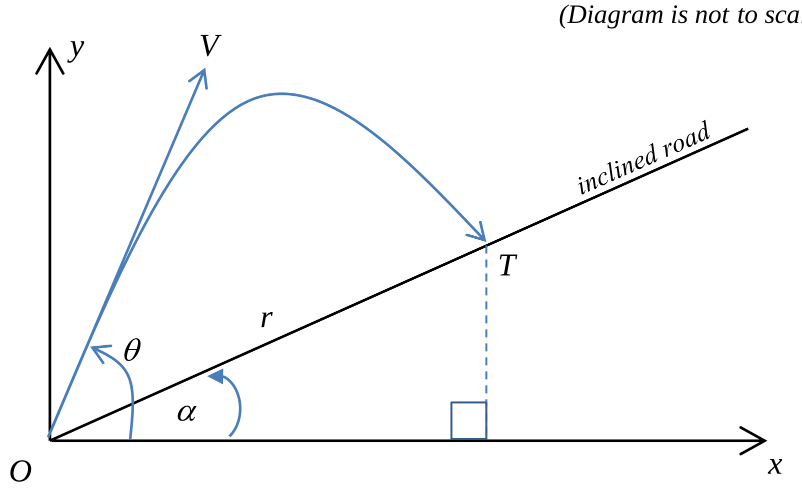
i) Show that the tangents at *P* and *Q* meet at *T*, where *T* is the point $(a(p+q), apq)$. **(2)**

ii) As *P* moves, *Q* is chosen such that $\angle POQ$ is always 90°, where *O* is the origin. Find the locus of *T*. **(2)**

Question 14 continues on the next page

c) The diagram shows an inclined road which makes an angle of α with the horizontal.

(Diagram is not to scale)



A projectile is fired from O, at the bottom of the inclined road, with a speed of V m/s at an angle of elevation θ to the horizontal as shown above. Using the axes above, you may assume that the position of the projectile is given by

$$x = Vt\cos\theta \quad \text{and} \quad y = Vt\sin\theta - \frac{1}{2}gt^2$$

where t is the time, in seconds, after firing, and g is the acceleration due to gravity. For simplicity, assume that $\dfrac{2V^2}{g} = 1$.

i) Show that the path of the trajectory of the projectile is $y = x\tan\theta - x^2\sec^2\theta$. **(2)**

ii) Show that the range of the projectile r = OT metres, up the inclined road is given by

$$r = \frac{\sin(\theta - \alpha)\cos\theta}{\cos^2\alpha} \ .$$ **(2)**

Student Number:___________________
Name:__________________

SECTION I Extension 1 Multiple Choice Answer Sheet 10 Marks

This sheet must be detached and handed in separately

Shade the correct answer:

1.	A ⬭	B ⬭	C ⬭	D ⬭
2.	A ⬭	B ⬭	C ⬭	D ⬭
3.	A ⬭	B ⬭	C ⬭	D ⬭
4.	A ⬭	B ⬭	C ⬭	D ⬭
5.	A ⬭	B ⬭	C ⬭	D ⬭
6.	A ⬭	B ⬭	C ⬭	D ⬭
7.	A ⬭	B ⬭	C ⬭	D ⬭
8.	A ⬭	B ⬭	C ⬭	D ⬭
9.	A ⬭	B ⬭	C ⬭	D ⬭
10.	A ⬭	B ⬭	C ⬭	D ⬭

SECTION 1:

1) $x = 3\cos 2t$

$\dot{x} = -6\sin 2t$

$\therefore$ greatest speed $= 6$ **(D)**

2) **(A)**

3) $\frac{d}{dx}\left(\sec\frac{x}{2}\right)$

$= \frac{1}{2}\sec\frac{x}{2}\tan\frac{x}{2}$ **(C)**

4) Centre when acc. is zero

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 - 2x$

$0 = 2 - 2x$

$x = 1$

$\therefore$ **(C)**

5) range: $0 \le y \le \pi$

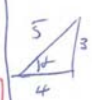
domain: $-1 \le \frac{3x}{2} \le 1$

$-2 \le 3x \le 2$

$-\frac{2}{3} \le x \le \frac{2}{3}$

(A)

6) Vertical component is



$v\sin\alpha = 30 \times \frac{3}{5}$

$= 18\text{ m/s}$ **(A)**

7) $y = e^{x+2}$

$x = e^{y+2}$

$\ln x = y + 2$

$y = \ln x - 2$ **(C)**

8) **(A)**

9) **(D)**

10) $\cos^2 2x = \frac{1}{2} + \frac{1}{2}\cos 4x$

$\therefore \int \frac{1}{2} + \frac{1}{2}\cos 4x \, dx$

$= \frac{x}{2} + \frac{\sin 4x}{8} + C$

(C)

DACCA
ACADC

QUESTION 11:

a) $\alpha + \beta + \gamma = -\frac{b}{a}$

$= -\frac{3}{2}$

(b) $\frac{d}{dx}\left(\tan^{-1} 2x^2\right)$

$= \frac{1}{1 + (2x^2)^2} \times 4x$

$= \frac{4x}{1 + 4x^4}$

c) $P(-2, 3)$ & $K(10, 1)$

externally in ratio $4:-3$

Let $J = (x, y)$

$x = \frac{-3(-2) + 4(10)}{4 - 3}$

$= 46$

$y = \frac{-3(3) + 4(1)}{4 - 3}$

$= -5$

$\therefore J = (46, -5)$

d) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\tan\theta = \left|\frac{1 - -\frac{1}{\sqrt{3}}}{1 + (1)\times(-\frac{1}{\sqrt{3}})}\right|$

$\tan\theta = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}$

$\theta = 75^\circ$

e) $\frac{5}{x+2} \le 1 \quad \ldots x \ne -2$

$\frac{5}{x+2} \times (x+2)^2 \le 1 \times (x+2)^2 \quad \ldots x \ne -2$

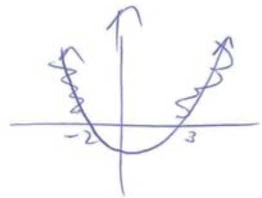
$5(x+2) \le (x+2)^2 \quad \ldots x \ne -2$

$(x+2)^2 - 5(x+2) \ge 0 \quad \ldots x \ne -2$

$(x+2)(x+2-5) \ge 0 \quad \ldots x \ne -2$

$(x+2)(x-3) \ge 0 \quad \ldots x \ne -2$

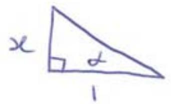
$\therefore x < -2$ or $x \ge 3 \quad \ldots x \ne -2$



(f) $\sin(\tan^{-1}x)$

let $\tan^{-1}x = \theta$

$\therefore \quad x = \tan\theta$



using pythagoras,
hypotenuse $= \sqrt{x^2+1}$

$\therefore \sin\theta = \dfrac{x}{\sqrt{x^2+1}}$

$\therefore \sin(\tan^{-1}x) = \dfrac{x}{\sqrt{x^2+1}}$

g) $\sin 2x = 2\sin^2 x$

$2\sin x\cos x = 2\sin^2 x$

$2\sin^2 x - 2\sin x\cos x = 0$

$2\sin x(\sin x - \cos x) = 0$

$\therefore 2\sin x = 0$ OR $\sin x = \cos x$

$\sin x = 0$ OR $\tan x = 1$

$\therefore x = 0, \pi, 2\pi, \frac{\pi}{4}, \frac{5\pi}{4}$

QUESTION 12

a) $\angle XZY = 90^\circ$ (angle in semi circle)

$\angle$ AB $= 90^\circ$ (exterior $\angle$ of cyclic quad.)

b) $\displaystyle\int_0^2 \frac{4x}{\sqrt{4x+1}}\,dx$

$x = \frac{1}{4}(u-1)$

$4x = u-1$

$u = 4x+1$

$du = 4dx$

$x=0$: $0 = \frac{1}{4}(u-1)$, $u=1$

$x=2$: $2 = \frac{1}{4}(u-1)$, $u=9$

$= \frac{1}{4}\displaystyle\int \frac{u-1}{\sqrt{u}}\,du$

$= \frac{1}{4}\displaystyle\int_1^9 u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du$

$= \frac{1}{4}\left[\frac{2}{3}u^{3/2} - 2u^{\frac{1}{2}}\right]_1^9$

$= \frac{1}{4}\left[\left(\frac{2}{3}(9)^{3/2} - 2(9)^{\frac{1}{2}}\right) - \left(\frac{2}{3}(1)^{3/2} - 2(1)^{\frac{1}{2}}\right)\right]$

$= \frac{1}{4}\left[(18-6) - (2/3 - 2)\right]$

$= \frac{1}{4}\left[12 + \frac{4}{3}\right] = \frac{10}{3}$

c) $f(x) = e^x - 3x$ & $f'(x) = e^x - 3$

$f(1) = e - 3$

$= -0.2817\ldots < 0$

$f(1.7) = e^{1.7} - 3\times1.7$

$= 0.3739\ldots > 0$

change of sign between $x=1$ & $x=1.7$

$\therefore$ root lies between.

$x_1 = 1.7 - \dfrac{e^{1.7} - 3(1.7)}{e^{1.7} - 3}$

$x_1 = 1.54\ldots$

$\therefore x_1 \doteqdot 1.55$ (3 s.f.)

$\therefore$ an approx. to the root is $x \doteqdot 1.55$.

d) $P(x) = x^3 + ax^2 + bx + c$

$P(0) = 0 = c$

$\therefore \boxed{c = 0}$

$P(3) = 0 = 27 + 9a + 3b$ ①

$P(-3) = 0 = -27 + 9a - 3b$ ②

$18a = 0$

$\boxed{a = 0}$

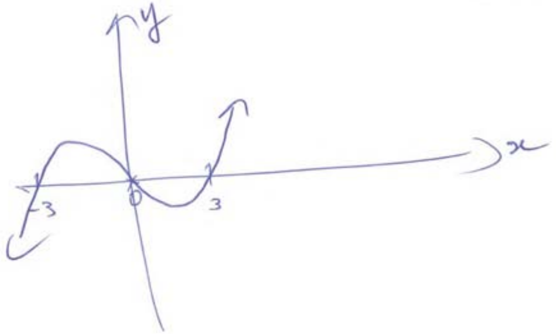
subst $a = 0$ into ①: $27 + 0 + 3b = 0$

$\boxed{b = -9}$

$\therefore P(x) = x^3 - 9x$

$P(x) = x(x^2 - 9)$

$= x(x-3)(x+3)$



e) $\therefore 5^n - 1$

Prove for $n = 1$: $5^1 - 1 = 4$

$\therefore$ true for $n = 1$

Assume true for $n = k$: where $k \in \mathbb{N}$

$5^k - 1$ is divisible by 4

$\therefore$ Let $5^k - 1 = 4P$ where $P \in \mathbb{N}$

Prove for $n = k+1$:

$5^{k+1} - 1 = 5^k \times 5 - 1$

$= (4P+1) \times 5 - 1$

$= 20P + 5 - 1$

$= 20P + 4$

$= 4(5P+1)$

$= 4M$ ... where $M = 5P+1$

$\therefore 5^{k+1} - 1$ is divisible by 4

QUESTION 13:

a) $\cos x - \sin x = A\cos x \cos\alpha - A\sin x \sin\alpha$.

Equating: $A\cos\alpha = 1$ ... (1)

$A\sin\alpha = 1$ ... (2)

Squaring & adding (1)&(2): $A^2\cos^2\alpha + A^2\sin^2\alpha = 2$

$A^2(\cos^2\alpha + \sin^2\alpha) = 2$

$A^2 = 2$

$A = \sqrt{2}$

From (1) & (2): $\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$ ... only 1st quad since $\cos\alpha$ & $\sin\alpha$ both > 0

$\therefore \cos x - \sin x = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$

b) $\cos x - \sin x = \frac{1}{\sqrt{2}}$

$\sqrt{2}\cos\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

$\cos\left(x + \frac{\pi}{4}\right) = \frac{1}{2}$

$x + \frac{\pi}{4} = \frac{\pi}{3} + 2n\pi$ OR $x + \frac{\pi}{4} = -\frac{\pi}{3} + 2n\pi$

$x = \frac{\pi}{12} + 2n\pi$ OR $= \frac{-7\pi}{12} + 2n\pi$

$\therefore$ general sol'n is

$x = \frac{\pi}{12} + 2n\pi$ OR $x = \frac{-7\pi}{12} + 2n\pi$

(7)

c) $\frac{dM}{dt} = -k(M - 20)$

$LHS = Ae^{-kt} \times -k$

$= -kAe^{-kt}$

$RHS = -k\left(20 + Ae^{-kt} - 20\right)$

$= -kAe^{-kt}$

$= LHS$

ii) when $t = 0$, $M = 160$

$160 = 20 + Ae^0$

$A = 140$

$\therefore$ $M = 20 + 140e^{-kt}$

when $t = 5$, $M = 100$

$100 = 20 + 140e^{-5k}$

$\frac{80}{140} = e^{-5k}$

$\ln\left(\frac{4}{7}\right) = -5k$

$\therefore k = \frac{\ln\left(\frac{4}{7}\right)}{-5}$ $(\doteq 0.11192)$

(iii) $M = 40$, $t = ?$

$M = 20 + 140e^{-kt}$

$40 = 20 + 140e^{-kt}$

$\frac{20}{140} = e^{-kt}$

$\ln\left(\frac{1}{7}\right) = -kt$

$\frac{-\ln 7}{-k} = t$

$t = 17.3866...$

$t \doteq 17$ min.

(8)

d)  $h = 2r$

$V = \pi r^2 h$

$V = \pi r^2 \times 2r$

$V = 2\pi r^3$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$\boxed{\frac{dV}{dt} = 6\pi r^2 \times \frac{dr}{dt}}$

$\frac{dV}{dt} = pA$

$= p\left(2\pi r^2 + 2\pi r h\right)$

$= p\left(2\pi r^2 + 2\pi \times r \times 2r\right)$

$\frac{dV}{dt} = p \times 6\pi r^2$

$\therefore 6\pi r^2 \times \frac{dr}{dt} = p \times 6\pi r^2$

$\frac{dr}{dt} = p$

ii) $r = \int \frac{dr}{dt}$

$= \int p \, dt$

$\therefore r = pt + c$

$t = 0, \ r = 0.5$

$\therefore 0.5 = p \times 0 + c$

$c = 0.5$

$\therefore \boxed{r = pt + 0.5}$

iii) $r = 0, \ t = 5$

$0 = p \times 5 + 0.5$

$p = -0.1$

QUESTION 14

a) i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 8x(x^2+4)$

$\frac{1}{2}v^2 = \int 8x^3 + 32x \, dx$

$\frac{1}{2}v^2 = 2x^4 + 16x^2 + C$

$v^2 = 4x^4 + 32x^2 + D$

when $x=0, \ v=8$: $64 = 0 + 0 + D$

$D = 64$

$\therefore v^2 = 4x^4 + 32x^2 + 64$

$= 4\left(x^4 + 8x^2 + 16\right)$

$v^2 = 4\left(x^2+4\right)^2$

$\therefore v = \pm 2\left(x^2+4\right)$

$\therefore \text{speed} = \left|\pm 2(x^2+4)\right|$

$S = 2\left(x^2+4\right)$ m/s

ii) velocity always +ve since it is initially 8 m/s & v needs to be zero before it can become −ve, but $2(x^2+4) > 0$ for all x.

iii) $v = 2(x^2+4)$

$$\frac{dx}{dt} = 2(x^2+4)$$

$$\frac{dt}{dx} = \frac{1}{2(x^2+4)}$$

$$t = \frac{1}{2}\int \frac{1}{x^2+4}\,dx$$

$$t = \frac{1}{2} \times \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$t = \frac{1}{4}\tan^{-1}\frac{x}{2} + C$$

when $t=0, x=0$:

$$0 = \frac{1}{4}\tan^{-1} 0 + C$$

$$\therefore C = 0$$

$$\therefore t = \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right)$$

when $x=2$: $t = \frac{1}{4}\tan^{-1}\left(\frac{2}{2}\right)$

$$t = \frac{1}{4}\tan^{-1} 1$$

$$t = \frac{1}{4} \times \frac{\pi}{4}$$

$$t = \frac{\pi}{16} \text{ sec.}$$

QUESTION 4:

a) Eq'n of tangents of P & Q respectively

$y = px - ap^2$ and $y = qx - aq^2$

Equating: $px - ap^2 = qx - aq^2$

$$x(p-q) = a(p^2-q^2)$$

$$x(p-q) = a(p-q)(p+q)$$

$$\therefore x = a(p+q)$$

Subst in $y = px - ap^2$

$$y = p \times a(p+q) - ap^2$$

$$y = ap^2 + apq - ap^2$$

$$y = apq$$

$$\therefore T = (a(p+q), apq)$$

ii) $\text{Grad}_{OP} \times \text{Grad}_{OQ} = -1$

$$\frac{ap^2}{2ap} \times \frac{aq^2}{2aq} = -1$$

$$\frac{pq}{4} = -1$$

$$\boxed{pq = -4}$$

Locus of T: $x = a(p+q)$ & $y = apq$

Subst $pq = -4$ into $y = apq$

$$y = a \times -4$$

$\boxed{y = -4a}$ .. locus of T

c) i) $x = Vt\cos\theta$ ... ①

$y = Vt\sin\theta - \frac{1}{2}gt^2$ ... ②

from ① $t = \frac{x}{V\cos\theta}$

Subst in ②:

$$y = V \times \frac{x}{V\cos\theta} \times \sin\theta - \frac{1}{2}g \times \frac{x^2}{V^2\cos^2\theta}$$

$$= x\tan\theta - x^2\sec^2\theta \times \frac{g}{2V^2}$$

$$= x\tan\theta - x^2\sec^2\theta \times 1 \quad \therefore \text{since } \frac{2V^2}{g} = 1$$

$$y = x\tan\theta - x^2\sec^2\theta$$

ii) Let $T = (x, y)$

then $x = r\cos\phi$

and $y = r\sin\phi$

From i): $y = x\tan\theta - x^2\sec^2\theta$

$$r\sin\phi = r\cos\phi \times \tan\theta - r^2\cos^2\phi \times \sec^2\theta$$

$$r\cos^2\phi\sec^2\theta = \cos\phi\tan\theta - \sin\phi$$

$$r = \frac{\cos\phi\tan\theta - \sin\phi}{\cos^2\phi\sec^2\theta}$$

$$= \frac{\cos\phi \times \frac{\sin\theta}{\cos\theta} - \sin\phi}{\frac{\cos^2\phi}{\cos^2\theta}}$$

$$= \frac{\frac{\cos\phi\sin\theta - \sin\phi\cos\theta}{\cos\theta}}{\frac{\cos^2\phi}{\cos^2\theta}}$$

$$= \frac{\cos\phi\sin\theta - \sin\phi\cos\theta}{\cos\theta} \times \frac{\cos^2\theta}{\cos^2\phi}$$

$$\therefore r = \frac{\sin(\theta - \phi) \times \cos\theta}{\cos^2\phi}$$